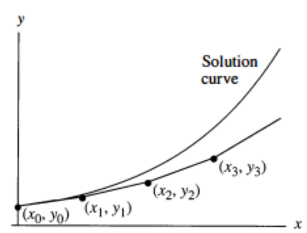
2.4: Numerical Approximation - Euler's Method

In Chapter 1, we saw a number of examples where the general first-order differential equation $y' = f(x, y)$ can be solved explicitly or exactly. However, this is the exception rather than the rule. For instance, the differential equation

$$\frac{dy}{dx} = e^{-x^2}$$

can only be solved with nonelementary functions and therefore we need a way of approximating the solution. Thus all the techniques from Chapter 1 are doomed to fail.

Example 1.



We wish to approximate the solution curve to $y' = d^{-x^2}$. We start with the initial point (x_0, y_0) and draw a small line segment of step size $\Delta x = h$ to get to the point (x_1, y_1) and repeat. You can see that our approximating is not great at first, but as $h \to 0$ we will get a good approximation of the solution curve. This is precisely the strategy of Euler's Method.

Euler's Method We start with the initial point (x_0, y_0) and solve for $y'(x_0, y_0) = f(x_0, y_0)$. We then repeat to get

$$\begin{aligned} x_1 &= x_0 + h & y_1 &= y_0 + h \cdot f(x_0, y_0) \\ x_2 &= x_1 + h & y_2 &= y_1 + h \cdot f(x_1, y_1) \\ x_3 &= x_2 + h & y_3 &= y_2 + h \cdot f(x_2, y_2) \\ x_4 &= x_3 + h & y_4 &= y_3 + h \cdot f(x_3, y_3) \\ &\vdots & &\vdots \end{aligned}$$

As an algorithm, with the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0,$$

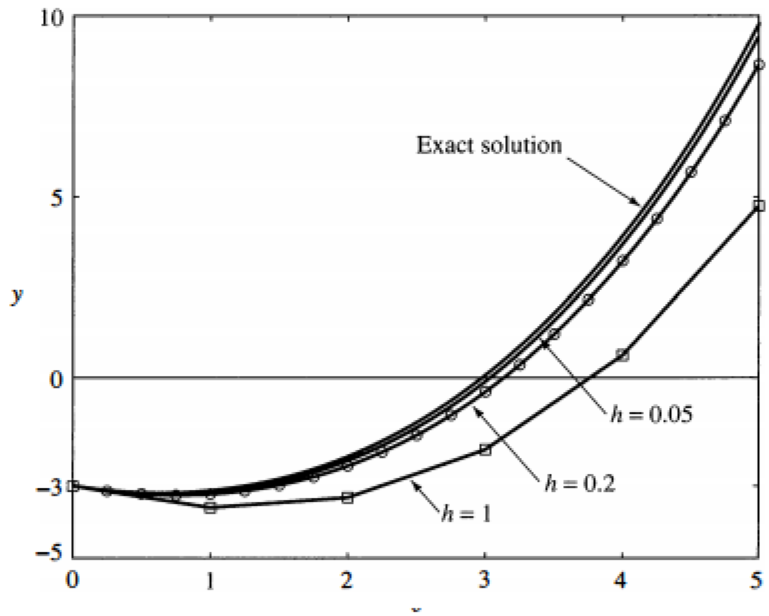
we get the algorithm for Euler's Method

$$x_{n+1} = x_n + h = x_0 + h \cdot (n+1), \quad y_{n+1} = y_n + h \cdot f(x_n, y_n) \quad (n \geq 0).$$

Example 2. Use Euler's Method to approximate the solution of the initial value problem

$$\frac{dy}{dx} = x + \tfrac{1}{5}y, \quad y(0) = -3$$

first with step size $h = 1$ on the interval [0,5] then with step size $h = 0.2$ on the interval [0,1].



Homework. 1-5, 11-15, 25 (odd)